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Weierstraß semigroups and nodal curves of type p, q

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Abstract

We study plane curves of type p, q having only nodes as singularities. Every Weierstraß semigroup is the Weierstraß semigroup of such a curve at its place at infinity for properly chosen p, q. We construct plane curves of type p, q with explicitly given nodes and determine their Weierstraß semigroups. Many such semigroups are found.

Introduction

Let $\mathcal{R}$ be a smooth projective algebraic curve of genus g defined over an algebraically closed field K of characteristic 0. For a closed point $P \in \mathcal{R}$ consider the meromorphic functions on $\mathcal{R}$ which have a pole at most at P, hence are regular everywhere else. By the Weierstraß Gap Theorem the set $H(P)$ of pole orders of such functions is a numerical semigroup of genus g, i.e. it is closed under addition and the *set of gaps* $\mathbb{N} \setminus H(P)$ consists of exactly g integers. Moreover for all but finitely many $P \in \mathcal{R}$ the gaps are $1, \dots, g$. The semigroup $H(P)$ is called the *Weierstraß semigroup* of $\mathcal{R}$ at P.

Hurwitz [Hu] asked in 1893: Which numerical semigroups do occur as Weierstraß semigroups?

Since then many classes of Weierstraß semigroups have been specified. For example all numerical semigroups of genus ≤ 8 are Weierstraß semigroups ([Ko1], [Ko-O]), as are complete intersection semigroups ([Pi]). The first correct example of a numerical semigroup which is not a Weierstraß semigroup was found by Buchweitz [B] in 1980. Such semigroups are now called *Buchweitz semigroups*. K.O. Stöhr and F. Torres ([T], Scholium 3.5) have shown that for any $g \geq 100$ *symmetric* Buchweitz semigroups of genus g exist, and in [Ko2] many Buchweitz semigroups are constructed.

In this paper we investigate Weierstraß semigroups using plane curves of type p, q. For relatively prime integers p, q with $1 < p < q$ these are the curves C in $\mathbb{A}^2(K)$ defined by a *Weierstraß equation of type* p, q

$$C : Y^p + aX^q + \sum_{\nu p + \mu q < pq} a_{\nu\mu} X^\nu Y^\mu = 0 \qquad (a_{\nu\mu} \in K, a \in K \setminus \{0\}).$$

Such curves are irreducible and have only one place P at infinity which is a point on the normalization $\mathcal{R}$ of the projective closure of C. We call $H(P)$ also the *Weierstraß semigroup of* C. It contains the semigroup $H_{pq} := \langle p, q \rangle$ generated by p and q as a subsemigroup. The

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method to study compact Riemann surfaces by investigating their plane models of type p, q can be attributed to Weierstraß (see the remark on page 544 of [HL]).

The gaps of $H(P)$ can be described by the holomorphic differentials ω on $\mathcal{R}$:

$$\mathbb{N} \setminus H(P) = \{\mathrm{ord}_P(\omega) + 1 \,|\, \omega \text{ holomorphic on } \mathcal{R}\}$$

where ord_P is the normed discrete valuation at the point P. The holomorphic differentials on $\mathcal{R}$ are closely connected to the adjoint curves of C. If C has only nodes as singularities, i.e. singularities of multiplicity 2 with two different tangents, these are the curves passing through all of the nodes. If the nodes are explicitly known the holomorphic differentials on $\mathcal{R}$ can be computed (Proposition 4.2), and so can be the Weierstraß semigroups $H(P)$. For this reason we are mainly interested in curves of type p, q having only nodes as singularities (at finite distance). We call them *nodal curves of type* p, q. It can be shown that any Weierstraß semigroup is the Weierstraß semigroup of a nodal curve of type p, q for properly chosen integers p and q (Theorem 6.4).

Given a nodal curve C of type p, q with explicitly known nodes there is a construction to eliminate some of its nodes. More precisely this means that for certain subsets $\{P_1, \dots, P_l\}$ of the set of nodes of C another curve C' of type p, q can be found having $\{P_1, \dots, P_l\}$ as its set of nodes (Corollary 3.2). By elimination of nodes a great number of Weierstraß semigroups and explicitly given nodal curves of type p, q having these Weierstraß semigroups can be specified, see Section 5.

In our considerations the numerical semigroups containing p and q play a crucial role. We begin with a geometric illustration of such semigroups.

1 An illustration of numerical semigroups

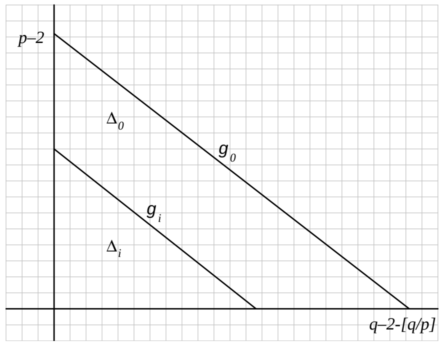
Let $p, q \in \mathbb{N}$ be relatively prime integers with $1 < p < q$, and let $H_{pq} = \langle p, q \rangle$ be the numerical semigroup generated by p and q. It is a symmetric semigroup with the conductor $c := (p-1)(q-1)$ and $d := {}^{c}/_{2}$ gaps $\gamma_1 < \cdots < \gamma_d$ where $\gamma_i = c - 1 - (a_i p + b_i q)$ with uniquely determined $a_i, b_i \in \mathbb{N}$. Therefore the gaps are in one-to-one correspondence with the d lattice points $(a, b) \in \mathbb{N}^2$ below the line

$$g_0 : pX + qY = c - 1.$$

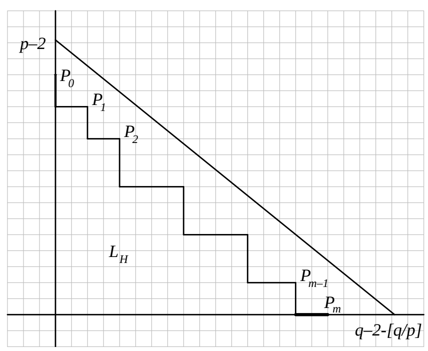
Let Δ_0 be the triangle which is determined by g_0 and the coordinate axes. On each parallel

$$g_i : pX + qY = c - 1 - i \qquad (i = 1, \dots, c - 1)$$

there is at most one lattice point, and it corresponds to the gap i of H_{pq}. Let Δ_i be the triangle determined by g_i and the coordinate axes.



A numerical semigroup H with $p, q \in H$ arises from H_{pq} by closing some of its gaps. If $(a,b) \in \Delta_1$ corresponds to such a gap, then also the lattice points of the rectangle $R_{a,b}$ with the corners $(0,0),(a,0),(a,b),(0,b)$ belong to gaps which are closed in H. Therefore these gaps correspond to a set $L_H \subset \Delta_1$ of lattice points determined by a lattice path starting on the Y-axis and ending on the X-axis with downward and right steps, see the following figure, where P_0 and P_1 have the same Y-coordinates if the first step in the lattice path is a right step, and P_{m-1} and P_m may have the same X-coordinates.



If $P_i = (\alpha_i, \beta_i)$ $(i = 0, \dots, m)$, then

$$\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_{m-1} \leq \alpha_m \leq q - 2 - \left\lfloor \frac{q}{p} \right\rfloor$$

$$p - 2 \geq \beta_0 \geq \beta_1 > \cdots > \beta_{m-1} > \beta_m = 0.$$

The numbers $c - 1 - (\alpha_i p + \beta_i q)$ $(i = 0, \dots, m)$ together with p and q form a system of generators of H.

We write $L = (P_0, \dots, P_m)$ for such subsets of lattice points in Δ_1. Not every $L = (P_0, \dots, P_m)$ belongs to a numerical semigroup. For two gaps γ, γ' of H_{pq} which are to be closed in H also $\gamma + \gamma'$ must be such a gap or an element of H_{pq}.

Examples 1.1. a) Let $P = (a,b) \in g_1,\ a \neq 0, b \neq 0$. Then the set $L = ((0,b), P, (a,0))$ does not correspond to a numerical semigroup, since with 1 all gaps of H_{pq} are closed in H.
b) If L consists of all lattice points in Δ_i, then $L = L_H$ where H is the semigroup obtained from H_{pq} by closing all its gaps which are $\geq i$.

Lemma 1.2. *Let γ and γ' be gaps of H_{pq} belonging to $(a,b) \in \Delta_1$ and $(a',b') \in \Delta_1$ respectively. Then $\gamma + \gamma'$ is a gap of H_{pq} if and only if $a + a' \geq q - 1$ or $b + b' \geq p - 1$. The gap $\gamma + \gamma'$ corresponds to $Q := (a,b) + (a',b') + (1-q,1)$ if $a + a' \geq q - 1$, and to $P := (a,b) + (a',b') + (1,1-p)$ if $b + b' \geq p - 1$.*

Proof. We have

$$\gamma + \gamma' = c - 1 + pq - p - q - [(a+a')p + (b+b')q] \geq 0$$

Write

$$\gamma + \gamma' = c - 1 - [(a + a' + 1)p + (b + b' - p + 1)q] \tag{1}$$

$$\gamma + \gamma' = c - 1 - [(a + a' - q + 1)p + (b + b' + 1)q]. \tag{2}$$

If $a + a' \geq q - 1$ or $b + b' \geq p - 1$, then $\gamma + \gamma'$ is a gap of H_{pq}, and it belongs to $Q \in \Delta_1$ or $P \in \Delta_1$ respectively.

Conversely let $\gamma + \gamma'$ be a gap of H_{pq} belonging to $(a'', b'') \in \Delta_1$. Then

$$(a + a' - q + 1)p + (b + b' + 1)q = a''p + b''q. \tag{3}$$

This implies

$$q \,|\, a'' - a - a' - 1 \text{ and } p \,|\, b'' - b - b' - 1.$$

As $a, a', a'' < q - 1$ and $b, b', b'' < p - 1$ we obtain

$$a'' - a - a' - 1 \in \{0, -q\} \text{ and } b'' - b - b' - 1 \in \{0, -p\}.$$

But $(a'', b'') \neq (a + a' + 1, b + b' + 1)$ by (3). Hence

$$a + a' + 1 - q = a'' \geq 0 \text{ or } b + b' + 1 - p = b'' \geq 0$$

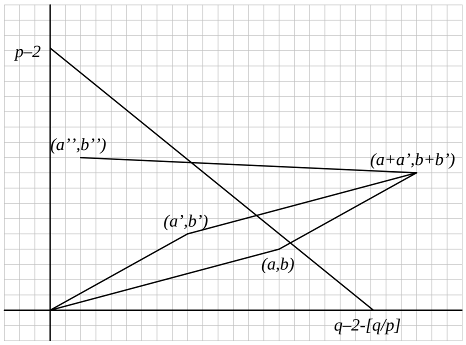
and $a + a' \geq q - 1$ or $b + b' \geq p - 1$. □

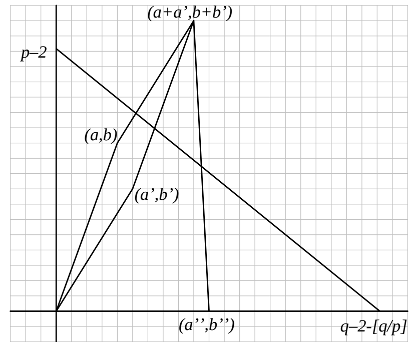
We obtain

Proposition 1.3. *$L = (P_0, \ldots, P_m)$ belongs to a numerical semigroup if and only if L is closed under the operations*

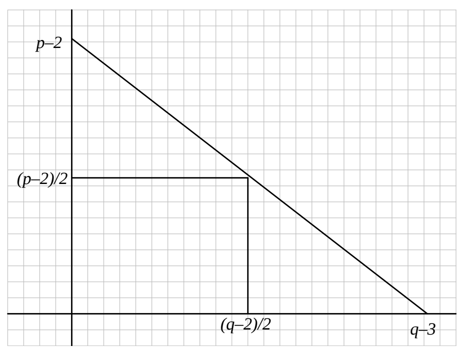
$$(a,b),(a',b') \mapsto \begin{cases} (a,b) + (a',b') + (1-q,1), \text{if } a + a' \geq q - 1, \\ (a,b) + (a',b') + (1,1-p), \text{if } b + b' \geq p - 1. \end{cases}$$

If $P_i = (\alpha_i, \beta_i)$ $(i = 0, \ldots, m)$, then it suffices to check the conditions for the (α_i, β_i) and (α_j, β_j) with $\alpha_i + \alpha_j \geq q - 1$ or $\beta_i + \beta_j \geq p - 1$.





Examples 1.4. a) A rectangle $R_{a,b}$ as above defines a numerical semigroup H if and only if $a < {}^1/_2(q-1), b < {}^1/_2(p-1)$. Then $H = \langle p, q, c-1-(ap+bq)\rangle =: H_{(a,b)}$. If the conditions are satisfied and L is defined by a lattice path inside the rectangle, then $L = L_H$ with a numerical semigroup $H \subset H_{(a,b)}$, since the closedness condition of the proposition is clearly fulfilled.



b) Let $\Delta_k \subset L \subset \Delta_i$ with $2i \geq k$ and $i \geq 1$. Then $L = L_H$ with a numerical semigroup H. In fact, let $(a,b),(a',b') \in L$ and $a + a' \geq q-1$. Then from $ap + bq \leq c-1-i$ and $a'p + b'q \leq c-1-i$ we obtain

$$(a + a' + 1 - q)p + (b + b' + 1)q \leq c - 1 - 2i \leq c - 1 - k$$

hence $(a,b) + (a',b') + (1-q,1) \in \Delta_k \subset L$. Similarly if $b + b' \geq p-1$.
c) For $r \in \mathbb{N}$ with $0 \leq r \leq p-2$ let L be the triangle with the corners $(0,0),(r,0),(0,r)$. Then $L = L_H$ with a numerical semigroup H. In fact, if $(a,b),(a',b') \in L$, then $a+b \leq r, a'+b' \leq r$. If $a + a' \geq q-1$, then we have for $(a'',b'') := (a,b) + (a',b') + (1-q,1)$ that

$$a'' + b'' = a + a' - q + b + b' + 2 \leq 2r - (q-2) \leq r.$$

Similarly, if $b + b' \geq p-1$.
d) Let $q = p+1$ and $H_p := H_{pq}$. In this case the gaps of H_p are in one-to-one correspondence with the $\binom{p}{2} = {}^c/_2$ lattice points of the triangle Δ with the corners $(0,0),(p-2,0),(0,p-2)$, that is with the $(a,b) \in \mathbb{N}^2$ with $a + b < p-1$. Let s_i be the section from $(0,p-2-i)$ to $(p-2-i,0)$ $(i = 0,\dots,p-2)$. The lattice points on this section correspond to the gaps

$ip+i+1,\dots,ip+p-1$ of H_p. Let Δ'_i be the triangle with the corners $(0,0),(p-2-i,0),(0,p-2-i)$ and let $\Delta'_k \subset L \subset \Delta'_i$ with $2i \geq k$, then L defines a numerical semigroup because the conditions of Proposition 1.3 are fulfilled. Especially this is true if we add to the lattice points of Δ'_k $(k \geq 2)$ some lattice points of s_{k-1}.

It is a much more difficult problem to decide which L define Weierstraß semigroups. Later in the paper we will describe classes of such L for which this is the case.

On the other hand, take for instance $p \geq 9, q = p+1$. By 1.4d) the numbers $1,\dots,p-1, 2p-7, 2p-5, 2p-2, 2p-1$ are the gaps of a numerical semigroup H of genus $g = p+3$ with $p, p+1 \in H$. Similarly for the numbers $1,\dots,p-1,2p-7,2p-6,2p-3,2p-1$. Let $l_2(H)$ denote the number of sums of two gaps of H. If $p \geq 13$, then in both cases $l_2(H) = 3p+7 > 3g-3$, hence the necessary condition for Weierstraß semigroups given by Buchweitz

$$l_2(H) \leq 3g - 3 \tag{4}$$

is hurt. For larger p there is an increasing number of possibilities to construct Buchweitz semigroups. The work by Komeda [Ko1] gives all semigroups of genus 16 and 17 for which (4) is hurt and contains a list with the numbers of all semigroups of genus ≤ 37 and with the numbers of those for which (4) is hurt. For $p = 13$ the two examples above are the semigroups of genus 16 for which (4) is hurt, the first one is the example found by Buchweitz [B] of a numerical semigroup which is not a Weierstraß semigroup.

2 Plane algebraic curves of type p, q

For relatively prime numbers $p, q \in \mathbb{N}$ with $1 < p < q$ let $C : F = 0$ be a curve of type p, q in $\mathbb{A}^2(K)$ with defining Weierstraß polynomial

$$F := Y^p + aX^q + \sum_{\nu p + \mu q < pq} a_{\nu\mu} X^\nu Y^\mu.$$

Proposition 2.1. *C is irreducible and has only one place P at infinity, hence also only one point at infinity. If $x, y \in K[C] = K[X,Y]/(F)$ are the residue classes of X, Y, then*

$$\mathrm{ord}_P(x) = -p,\ \mathrm{ord}_P(y) = -q.$$

Proof. ([Kn], Satz 1) □

Proposition 2.2. *Let $\bar{C}$ be the projective closure of C and Q its point at infinity. At Q the curve $\bar{C}$ has multiplicity $q-p$ and singularity degree*

$$\delta_Q = \frac{1}{2}(q-p-1)(q-1).$$

Proof. ([Kn], Satz 2) □

Corollary 2.3. *If C has l nodes and no other singularities (at finite distance), then the normalization $\mathcal{R}$ of $\bar{C}$ has genus*

$$g = \frac{1}{2}(p-1)(q-1) - l.$$

Proof. This follows from the genus formula

$$g = \frac{1}{2}(q-1)(q-2) - \sum_{R \in \tilde{C}(K)} \delta_R$$

for plane projective curves of degree q (see e.g. [Ku], Theorem 14.7) and the fact that $\delta_R = 1$ for a node R and $\delta_Q = {}^1\!/_2(q-p-1)(q-1)$ for the point Q at infinity. $\square$

By Proposition 2.1 the Weierstraß semigroup $H(P)$ of a curve C of type p, q contains p and q, hence we are in the situation of Section 1. If C is a nodal curve, then by Corollary 2.3 the number of its nodes is at most ${}^1\!/_2(p-1)(q-1)$.

Examples 2.4 (Lissajous curves)**.** The Chebyshev polynomials $T_n(X)$ of the first kind are defined by

$$T_0(X) := 1, \quad T_1(X) := X, \quad T_n(X) := 2XT_{n-1}(X) - T_{n-2}(X) \text{ for } n \geq 2.$$

They have degree n and leading coefficient 2^{n-1}. If p and q are relatively prime integers with $1 < p < q$, then the *Lissajous curve* C_L *of type* p, q is the plane curve defined over $\mathbb{C}$ by the polynomial

$$L_{p,q}(X,Y) := T_p(Y) - T_q(X).$$

Up to a factor 2^{p-1} it is a Weierstraß polynomial of type p, q. As $T_n(T_m(X)) = T_{nm}(X)$ the curve C_L has the parametric representation $x = T_p(t),\ y = T_q(t)$. It has the nodes

$$(x_k, y_l) := \left(\cos\frac{k\pi}{q}, \cos\frac{l\pi}{p}\right) \ (k = 1, \ldots, q-1, l = 1, \ldots, p-1, k \equiv l \bmod 2).$$

Their number is $d := {}^1\!/_2(p-1)(q-1)$.

In fact, we have $T'_p(Y) = pU_{p-1}(Y), T'_q(X) = qU_{q-1}(X)$ with the Chebyshev polynomials U_{p-1}, U_{q-1} of the second kind. Here U_{q-1} has simple zeros x_k $(k = 1, \ldots, q-1)$ and U_{p-1} has simple zeros y_l $(l = 1, \ldots, p-1)$ so that the singularities of C_L are found among the (x_k, y_l) $(k = 1, \ldots, q-1, l = 1, \ldots, p-1)$. The Hesse determinant of $L_{p,q}$ does not vanish at the (x_k, y_l) since the x_k, y_l are simple zeros of T'_q resp. T'_p. Therefore the singularities are nodes. Moreover it is known that

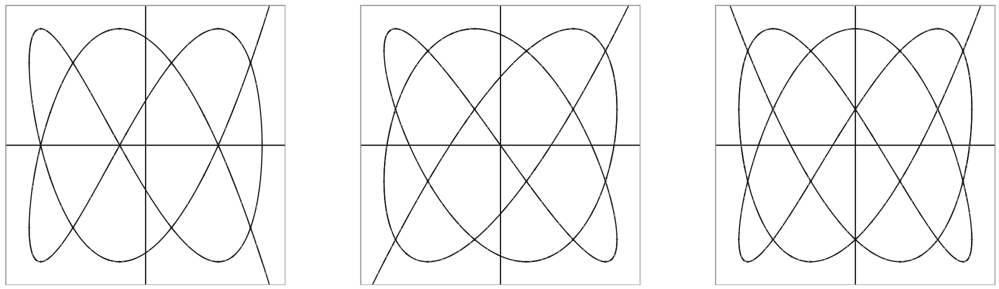
$$T_q\left(\cos\frac{k\pi}{q}\right) = \cos(k\pi) = (-1)^k, \quad T_p\left(\cos\frac{l\pi}{p}\right) = \cos(l\pi) = (-1)^l$$

and it follows that $L_{p,q}(x_k, y_l) = T_p(y_l) - T_q(x_k) = 0$ if and only if $k \equiv l \mod 2$.

The real points of the Lissajous curves defined by

$$\begin{aligned}
L_{4,7} &= 8Y^4 - 8Y^2 + 1 - 64X^7 + 112X^5 - 56X^3 + 7X,\\
L_{5,7} &= 16Y^5 - 20Y^3 + 5Y - 64X^7 + 112X^5 - 56X^3 + 7X \text{ and}\\
L_{5,8} &= 16Y^5 - 20Y^3 + 5Y - 128X^8 + 256X^6 - 160X^4 + 32X^2 - 1
\end{aligned}$$

are sketched in the following figures



The Weierstraß semigroup of C_L is $\mathbb{N}$. Since the $\cos {}^{k\pi}/_q$ $(k = 1, \ldots, q-1)$ are algebraic numbers the zero-set of $L_{p,q}(X,Y)$ in $\mathbb{A}^2(K)$ defines over any algebraically closed base field K of characteristic zero a nodal curve of type p, q with d distinct nodes.

Classical Lissajous curves play a role in physics in the theory of the pendulum. These curves are parametrized by the trigonometric sine function. The variation of the classical curves by using Chebyshev polynomials is introduced in [Pe].

3 Elimination of nodes

Let C be a curve of type p, q. Under certain assumptions the following procedure allows to eliminate singularities of C. In particular one can produce from a nodal curve C further nodal curves with fewer nodes. If the nodes of C are explicitly known the new curves also have explicitly known nodes. In this case one can compute their Weierstraß semigroups (see Proposition 4.2).

For relatively prime polynomials $G, H \in K[X,Y]\setminus\{0\}$ set $G_d := G + d \cdot H$ with $d \in K\setminus\{0\}$. Let $C := V(G)$, $D := V(H)$ and $C_d := V(G_d)$ be the corresponding schemes in $\mathbb{A}^2(K)$ and $\mathrm{Sing}(C)$, $\mathrm{Sing}(D)$, $\mathrm{Sing}(C_d)$ their sets of singularities. For a polynomial $\varphi \in K[X,Y]$ we denote its partial derivatives by φ_X and φ_Y.

Proposition 3.1. *Assume that* $\mathrm{Sing}(C_d)$ *is finite for all* $d \in K \setminus \{0\}$.
a) Then for all $d \in K \setminus \{0\}$

$$\mathrm{Sing}(C) \cap \mathrm{Sing}(D) \subset \mathrm{Sing}(C_d).$$

b) For almost all $d \in K \setminus \{0\}$ *we have equality in a).*

Proof. a) $\mathrm{Sing}(C_d)$ is the set of common zeros of

$$G_d = G + dH, (G_d)_X = G_X + dH_X \text{ and } (G_d)_Y = G_Y + dH_Y.$$

For all $d \neq d'$ from $K \setminus \{0\}$ we have

$$C_d \cap C_{d'} = C_d \cap C = C_d \cap D = C \cap D \tag{1}$$

since $(G_d, G_{d'}) = (G_d, G) = (G_d, H) = (G, H)$, and similarly

$$\mathrm{Sing}(C_d) \cap \mathrm{Sing}(C_{d'}) = \mathrm{Sing}(C_d) \cap \mathrm{Sing}(C) = \mathrm{Sing}(C_d) \cap \mathrm{Sing}(D) = \mathrm{Sing}(C) \cap \mathrm{Sing}(D). \tag{1'}$$

Moreover these sets are finite as G and H are relatively prime. In particular a) holds.

Assumption: $\mathrm{Sing}(C_d) \neq \mathrm{Sing}(C) \cap \mathrm{Sing}(D)$ for infinitely many $d \in K \setminus \{0\}$.

Then since $\mathrm{Sing}(C_d) \cap \mathrm{Sing}(C_{d'}) = \mathrm{Sing}(C) \cap \mathrm{Sing}(D)$ for $d \neq d'$ the set

$$S := \bigcup_{d \in K \setminus \{0\}} \mathrm{Sing}(C_d)$$

is infinite. On the other hand (1') and (1) imply that

$$(2) \quad S \cap \mathrm{Sing}(C) = S \cap \mathrm{Sing}(D) = \mathrm{Sing}(C) \cap \mathrm{Sing}(D) \text{ and } C_d \cap S \subset (C \cap D) \cup \mathrm{Sing}(C_d)$$

are finite sets for all $d \in K \setminus \{0\}$.

Let $\Gamma := G + Z \cdot H \in K[X, Y, Z]$. Then S is the set of closed points of the image of $V(\Gamma, \Gamma_X, \Gamma_Y) \setminus V(Z) \subset \mathbb{A}^3(K)$ by the orthogonal projection into the X, Y-plane. In particular S is an infinite constructible set in $\mathbb{A}^2(K)$.

For each $d \in K$ we have

$$G_X \cdot H - H_X \cdot G = H \cdot (G_d)_X - H_X \cdot G_d, \; G_Y \cdot H - H_Y \cdot G = H \cdot (G_d)_Y - H_Y \cdot G_d.$$

Therefore

$$(3) \qquad \text{Each } (\xi, \eta) \in S \text{ is a common zero of } G_X \cdot H - H_X \cdot G \text{ and } G_Y \cdot H - H_Y \cdot G.$$

These polynomials cannot be relatively prime since S is an infinite set. They have a common prime divisor P so that $S \cap V(P)$ is also infinite. This is an infinite constructible subset of the curve $V(P) \subset \mathbb{A}^2(K)$. It follows that $S \cap V(P)$ is open and dense in $V(P)$.

Now we distinguish three cases:

(i) If P divides H, then P divides $H_X \cdot G$ and P divides $H_Y \cdot G$. Since G and H are relatively prime P divides H_X, P divides H_Y and $V(P) \subset V(H, H_X, H_Y) = \mathrm{Sing}(D)$. This means that $S \cap \mathrm{Sing}(D)$ contains the infinite set $S \cap V(P)$ contradicting (2).

(ii) If P divides G, then we obtain a contradiction against the finiteness of $S \cap \mathrm{Sing}(C)$.

(iii) If P does not divide $G \cdot H$ the set $V(P) \cap V(G \cdot H)$ is finite and thus the set $U := (S \cap V(P)) \setminus V(G \cdot H)$ is open and dense in $V(P)$. The quotient G/H defines a non-vanishing regular function φ on U. By (3) we have

$$(G/H)_X(\xi, \eta) = (G/H)_Y(\xi, \eta) = 0 \text{ for all } (\xi, \eta) \in U.$$

Therefore the differential $d\varphi$ vanishes on U, and φ is constant on U and not zero. Hence there exists $d \in K \setminus \{0\}$ with $U \subset C_d$. But then since $U \subset S$ the set $C_d \cap S$ is infinite, again contradicting (2).

The above assumption is false, and we have shown that $\mathrm{Sing}(C_d) = \mathrm{Sing}(C) \cap \mathrm{Sing}(D)$ for almost all $d \in K \setminus \{0\}$. $\square$

Let $C : G = 0$ be a curve of type p, q, and let $\mathcal{F}$ denote the filtration of $K[X, Y]$ with $\deg_{\mathcal{F}}(X) = p$, $\deg_{\mathcal{F}}(Y) = q$ and $\deg_{\mathcal{F}}(a) = 0$. We set $G_d := G + d \cdot H$ where $H \in K[X, Y] \setminus \{0\}$ with $\deg_{\mathcal{F}}(H) < pq$ and $d \in K$. Then $G_d = 0$ defines a curve C_d of type p, q and $\mathrm{Sing}(C_d)$ is a finite set. Clearly G and H are relatively prime. Thus the preconditions of Proposition 3.1 are fulfilled.

Corollary 3.2. *If* $\operatorname{Sing}(C) \cap \operatorname{Sing}(D)$ *consists of nodes of* C*, then* C_d *is for almost all* $d \in K \setminus \{0\}$ *a nodal curve with the set of nodes* $\operatorname{Sing}(C) \cap \operatorname{Sing}(D)$.

Proof. From

$$(G_d)_{XX} = G_{XX} + dH_{XX},\ (G_d)_{XY} = G_{XY} + dH_{XY},\ (G_d)_{YY} = G_{YY} + dH_{YY}$$

it follows for the Hesse determinants Hess_G and $\operatorname{Hess}_{G_d}$ of G and G_d that

$$\text{(4)}\quad \operatorname{Hess}_{G_d} = \operatorname{Hess}_G + d\,(G_{XX}H_{YY} + G_{YY}H_{XX} - 2G_{XY}H_{XY}) + d^2\left(H_{XX}H_{YY} - {H_{XY}}^2\right).$$

If $\operatorname{Sing}(C) \cap \operatorname{Sing}(D)$ consists of the nodes (x_k, y_k) $(k = 1, \ldots, l)$ of C, then

$$\operatorname{Hess}_G(x_k, y_k) \neq 0\ (k = 1, \ldots, l).$$

It follows from Proposition 3.1 and from (4) that for almost all $d \in K \setminus \{0\}$ we have $\operatorname{Sing}(C_d) = \operatorname{Sing}(C) \cap \operatorname{Sing}(D)$ and $\operatorname{Hess}_{G_d}(x_k, y_k) \neq 0$ for $k = 1, \ldots, l$. Therefore the (x_k, y_k) are also nodes of C_d. □

Let C be a nodal curve of type p, q having certain nodes $(x_1, y_1), \ldots, (x_l, y_l)$. We want to obtain a nodal curve C_d such that the nodes $(x_1, y_1), \ldots, (x_l, y_l)$ are preserved while all other nodes of C are eliminated. According to the above construction we have to choose a curve $D : H = 0$ having the singularities (x_k, y_k) $(k = 1, \ldots, l)$ and being regular at the other nodes of C. If we can find such an H we say that C_d is obtained from C by *elimination of nodes.*

For example we can try $H = L^2$ with $L \in K[X, Y]$, $2\deg_{\mathcal{F}}(L) < pq$ where the curve $L = 0$ passes through the nodes (x_k, y_k) $(k = 1, \ldots, l)$, but through none of the other nodes of C. In particular the curve $L = 0$ can be a union of lines:

$$L = \prod_{i=1}^{r}(Y - m_iX + a_i)\prod_{j=1}^{s}(X - b_j)\ (m_i, a_i, b_j \in K) \text{ with } 2(pr + qs) < pq.$$

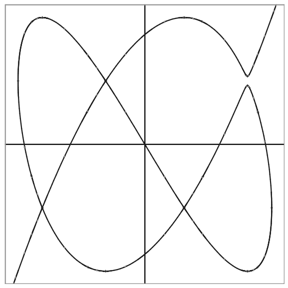
Example 3.3. Elimination of the nodes of the Lissajous curve of type $3, 5$

$$L_{3,5} = 4Y^3 - 3Y - 16X^5 + 20X^3 - 5X = 0.$$

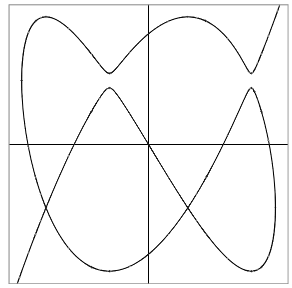
The figures show the real points of the curves $C_i : L_{3,5} + G_i = 0$ $(i = 1, \ldots, 4)$ where

$$\begin{aligned}
G_1 &= {}^1\!/_{50}\left(X - {}^1\!/_4\left(1 - \sqrt{5}\right)\right)^2 (Y + {}^1\!/_2)^2,\\
G_2 &= {}^1\!/_{50}\,(Y + {}^1\!/_2)^2,\\
G_3 &= {}^1\!/_{50}\left(X + {}^1\!/_4\left(1 - \sqrt{5}\right)\right)^2,\\
G_4 &= {}^1\!/_{50}.
\end{aligned}$$

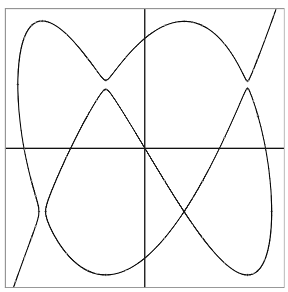
C_1:



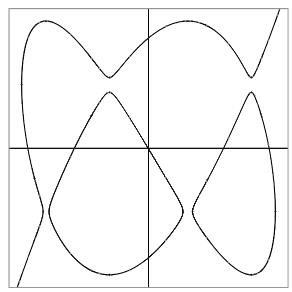
C_2:



C_3:



C_4:



Their Weierstraß semigroups are the numerical semigroups H with $3, 5 \in H$ except for $H = \mathbb{N}$.

If Γ is a nodal curve of type p, q whose nodes are also nodes of another such curve C, then Γ is obtained from C by elimination of nodes: If $C : Y^p - X^q + H_1 = 0$ and $\Gamma : Y^p - X^q + H_2 = 0$, take $H := H_2 - H_1$. Then $\Gamma = C_1 : G + 1 \cdot H = 0$.

Proposition 3.4. *If κ_1 and κ_2 are the sets of nodes of nodal curves of type p, q, then $\kappa_1 \cap \kappa_2$ is as well the set of nodes of such a curve.*

Proof. Let κ_i be the set of nodes of the nodal curve Γ_i of type p, q defined by $F_i = Y^p - X^q + H_i$ $(i = 1, 2)$. Set $G := F_1, H := H_2 - H_1, D := V(H)$. Then with the notation of 3.1 we have $C = \Gamma_1, C_1 = \Gamma_2$ and $\kappa_1 \cap \kappa_2 = \mathrm{Sing}(C) \cap \mathrm{Sing}(D)$ by (1'). Hence by 3.2 $C_d : F_1 + d \cdot H = 0$ is for almost all $d \in K \setminus \{0, 1\}$ a nodal curve with the set of nodes $\kappa_1 \cap \kappa_2$. □

Here is another class of nodal curves of type p, q.

Proposition 3.5. *For $l_1, l_2 \in \mathbb{N}$ with $1 \le l_1 \le q/2, 1 \le l_2 \le p/2$ let $x_0, \dots, x_{q-l_1-1} \in K$ resp. $y_0, \dots, y_{p-l_2-1} \in K$ be pairwise distinct. Set*

$$H(X) := \prod_{i=0}^{l_1-1} (X - x_i)^2 \prod_{i=l_1}^{q-l_1-1} (X - x_i),\ G(Y) := \prod_{j=0}^{l_2-1} (Y - y_j)^2 \prod_{j=l_2}^{p-l_2-1} (Y - y_j).$$

Then $C_d : G + d \cdot H = 0$ is for almost all $d \in K \setminus \{0\}$ a nodal curve of type p, q with set of nodes

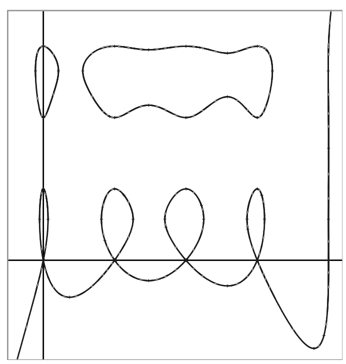
$$N := \{(x_i, y_j) \mid i = 0 \ldots l_1 - 1, j = 0, \ldots, l_2 - 1\}.$$

Proof. For all $d \in K \setminus \{0\}$ the curve C_d is of type p, q, and $\mathrm{Sing}(C_d)$ is finite. Obviously G and H are relatively prime. By Proposition 3.1 we have $\mathrm{Sing}(C_d) = \mathrm{Sing}(C) \cap \mathrm{Sing}(D)$ for almost all $d \in K \setminus \{0\}$ where $C := V(G), D := V(H)$. Since $(G_d)_X = d \cdot H'$ and $(G_d)_Y := G'$ it follows that $\mathrm{Sing}(C) \cap \mathrm{Sing}(D) = N$. The Hesse determinant $\mathrm{Hess}_{G_d}(X, Y) = d \cdot G''(X) \cdot H''(Y)$ does not vanish at the points of N, hence they are nodes of C_d. $\square$

The next figure shows the real points of the curve of type 5,9 with the equation

$$Y^2(Y-1)(Y-2)(Y-3) - X^2(X-1)^2(X-2)^2(X-3)^2(X-4) = 0$$

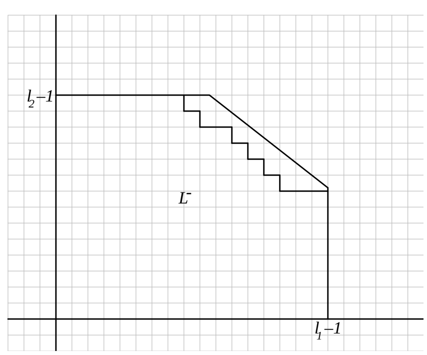
having 4 nodes on the X-axis.



Again for $l_1, l_2 \in \mathbb{N}$ with $1 \le l_1 \le q/2, 1 \le l_2 \le p/2$ let L be the set of lattice points in the rectangle R_{l_1-1, l_2-1} below the line $pX + qY = pq/2$. Set

$$E := \{(\alpha, \beta) \in L \mid \alpha < l_1 - 1, (\alpha + 1, \beta) \notin L\}.$$

and $L^- := L \setminus E$. The following figure shows L^- with $p = 43$, $q = 55$, $l_1 = 18$ and $l_2 = 15$.



With pairwise distinct $x_0,\dots,x_{l_1-1}\in K$ resp. $y_0,\dots,y_{l_2-1}\in K$ and a subset $\lambda\subset L^-$ let $\kappa(\lambda):=\{(x_m,y_n)\,|\,(m.n)\in\lambda\}$.

Proposition 3.6. *There exists a nodal curve of type p,q with the set of nodes $\kappa(\lambda)$.*

Proof. We apply elimination of nodes to a curve $C:F=0$ of type p,q with set of nodes $N=\{(x_m,y_n)\,|\,m=0,\dots,l_1-1,n=0\dots,l_2-1\}$, see Proposition 3.5. To this aim we construct a polynomial $H=\sum_{(i,j)\in L}u_{ij}X^iY^j\in K[X,Y]$ such that for $(m,n)\in R_{l_1-1,l_2-1}\cap\mathbb{N}^2=:R$

$$H(x_m,y_n)=0 \text{ if } (m,n)\in\lambda,\ \ H(x_m,y_n)\neq 0 \text{ if } (m,n)\notin\lambda.$$

We have $\deg_{\mathcal{F}}H<{}^{pq}/_2$, hence $C_d:F+d\cdot H^2=0$ is for almost all $d\in K\setminus\{0\}$ the desired curve, see Corollary 3.2.

For $M,N\subset R$ let $\Delta_{M,N}$ denote the matrix $\left({x_m}^i{y_n}^j\right)_{(m,n)\in M,(i,j)\in N}$ where the rows (columns) are ordered according to the following order of the points of R:

$$(0,0),(1,0),\dots,(l_1-1,0),(0,1),(1,1),\dots,(l_1-1,1),\dots,(0,l_2-1),\dots,(l_1-1,l_2-1).$$

Since L is given by a lattice path as in the second figure of Section 1 the matrix $\Delta_{L,L}$ has the form

$$\Delta_{L,L}=\begin{pmatrix} V_{0,0} & y_0V_{0,1} & {y_0}^2V_{0,2} & \dots & {y_0}^lV_{0,l}\\ V_{1,0} & y_1V_{1,1} & {y_1}^2V_{1,2} & \dots & {y_1}^lV_{1,l}\\ \vdots & \vdots & \vdots & \dots & \vdots\\ V_{l,0} & y_lV_{l,1} & {y_l}^2V_{l,2} & \dots & {y_l}^lV_{l,l}\end{pmatrix}$$

where $l:=l_2-1$ and the matrices $V_{i,j}$ depend only on the x_m. Here $V_{i,j+1}$ is obtained from $V_{i,j}$ by deleting some of its last columns, and $V_{i+1,j}$ by deleting the corresponding rows of $V_{i,j}$. The $V_{i,i}$ are van der Monde matrices. With elementary column transformations one shows that $\det(\Delta_{L,L})\neq 0$.

For $(\alpha,\beta)\in E$ and $k=1,\dots,l_1-1-\alpha$ set

$$L_{(\alpha,\,\beta,k)}:=(L\setminus\{(\alpha,\beta)\})\cup\{(\alpha+k,\beta)\}.$$

The matrix $\Delta_{L_{(\alpha,\,\beta,k)},L}$ is obtained from $\Delta_{L,L}$ if $V_{\beta,i}$ $(i=0,\dots,l)$ is replaced by the matrix $\tilde V_{\beta,i}$, where in the last row of $V_{\beta,i}$ the value x_α is everywhere replaced by $x_{\alpha+k}$. The $\tilde V_{i,i}$ $(i=0,\dots,l)$ are again van der Monde matrices. Therefore $\det(\Delta_{L_{(\alpha,\,\beta,k)},L})\neq 0$, too.

For any $(\alpha',\beta')\in R\setminus\lambda$ the matrix $\Delta_{\lambda\cup\{(\alpha',\,\beta')\},L}$ is a submatrix of $\Delta_{L,L}$, in case $(\alpha',\beta')\in L$, or of $\Delta_{L_{(\alpha,\,\beta',k)},L}$, in case $(\alpha',\beta')\notin L,\alpha'=\alpha+k$ with $(\alpha,\beta')\in E$. Thus in any case

$$\operatorname{rank}(\Delta_{\lambda\cup\{(\alpha',\,\beta')\},L})=\operatorname{rank}(\Delta_{\lambda,L})+1.$$

$\Delta_{\lambda,L}$ is the coefficient matrix of the linear system of equations for the unknowns u_{ij}

$$(S)\qquad \sum_{(i,j)\in L}u_{ij}{x_m}^i{y_n}^j=0\qquad((m,n)\in\lambda)$$

and $\Delta_{\lambda\cup\{(\alpha',\,\beta')\},L}$ of the system

$$(S_{(\alpha',\,\beta')})\qquad \sum_{(i,j)\in L}u_{ij}{x_m}^i{y_n}^j=0\qquad((m,n)\in\lambda\cup\{(\alpha',\beta')\}).$$

We find $u_{ij} \in K$ which solve (S) and do not solve $(S_{(\alpha',\beta')})$ for any $(\alpha',\beta') \in R \setminus \lambda$. Then $H = \sum_{(i,j)\in L} u_{ij} X^i Y^j$ is the polynomial we were looking for. $\square$

Remark 3.7. If we set $E' := \{(\alpha,\beta) \in L \mid \beta < l_2 - 1, (\alpha, \beta+1) \notin L\}$ and $L^{=} := L \setminus E'$, then Proposition 3.6 remains true with $L^{=}$ instead of L^{-}, due to the symmetry of the situation. In general $L^{=}$ is different from L^{-}, see the figure above.

4 Weierstraß semigroups and adjoints of nodal curves of type p, q

In the following let $C : F(X,Y) = 0$ be a curve of type p, q with place P at infinity, coordinate ring $K[C] = K[x,y]$, projective closure $\bar{C}$, and let $\mathcal{R}$ be the normalization of $\bar{C}$. Let $L := K(x,y)$ be the field of rational (meromorphic) functions of C over K, let $\Omega^1_{L/K}$ be its module of differentials and Ω the K-vector space of holomorphic differentials on $\mathcal{R}$. We shall use the fact that the gaps of the Weierstraß semigroup of C are the numbers $\mathrm{ord}_P(\omega) + 1$ for the $\omega \in \Omega$.

Since $F_Y(x,y) \neq 0$ every $\omega \in \Omega^1_{L/K}$ can be written

$$\omega = \frac{\Phi(x,y)}{F_Y(x,y)} dx \text{ with } \Phi(X,Y) \in K[X,Y].$$

By [G], proof of Theorem 12 we have $\omega \in \Omega$ if and only if the following two conditions are satisfied:
(1) $\mathrm{ord}_P(\omega) \geq 0$,
(2) $\Phi(x,y)$ is contained in the conductor from $K[\widetilde{C}]$ to $K[C]$ where $\widetilde{C}$ is the normalization of C.

We call the curves $\Phi = 0$ where Φ satisfies (2) the *adjoints* of C. If C has only the nodes $(x_1,y_1),\dots,(x_l,y_l)$ as singularities, then condition (2) is equivalent to
(2') $\Phi(x_i,y_i) = 0$ for $i = 1,\dots,l$.

For the moment we do not use the node condition. Denote by Ω_∞ the K-vector space of all $\omega \in \Omega^1_{L/K}$ of the form $\omega = {}^{\Phi(x,y)}/_{F_Y(x,y)} dx$ with a polynomial Φ, which satisfy (1). Let $\gamma_1 < \cdots < \gamma_d$ be the gaps of H_{pq}: $\gamma_i = (p-1)(q-1) - 1 - (a_i p + b_i q)$ with a unique $(a_i, b_i) \in \mathbb{N}^2$ $(i = 1,\dots,d)$.

Lemma 4.1. *The differentials*

$$\omega_i := \frac{x^{a_i} y^{b_i}}{F_Y(x,y)} dx \ (i = 1,\dots,d)$$

form a basis of Ω_∞.

Proof. We have

$$\begin{aligned}
\mathrm{ord}_P(\omega_i) &= -(a_i p + b_i q) - \mathrm{ord}_P(F_Y(x,y)) + \mathrm{ord}_P(dx) \\
&= -(a_i p + b_i q) + (p-1)q - (p+1) \\
&= (p-1)(q-1) - 1 - (a_i p + b_i q) - 1 = \gamma_i - 1 \geq 0.
\end{aligned}$$

Thus $\omega_i \in \Omega_\infty$ $(i = 1, \ldots, d)$.

Conversely let $\omega = {}^{\Phi(x,y)}/_{F_Y(x,y)}dx \in \Omega_\infty$. Since F is a monic polynomial in Y of degree p we can assume after reduction of Φ modulo F that $\deg(\Phi) \le p-1$. Write

$$\Phi(x,y) = \varphi_1(x)y^{p-1} + \cdots + \varphi_p(x) \quad (\varphi_j \in K[X]).$$

If $m \le p-1$, then different monomials of the form $x^n y^m$ have different pole orders $np+mq$ thanks to the fact that p and q are relatively prime. When $\lambda x^a y^b$ is the term of highest pole order of Φ, then

$$\mathrm{ord}_P(\omega) = -(ap+bq) + (p-1)q - (p+1) = (p-1)(q-1) - 2 - (ap+bq).$$

From $\mathrm{ord}_P(\omega) \ge 0$ follows that $ap+bq \le (p-1)(q-1)-2$. Then $(p-1)(q-1)-1-(ap+bq)$ is a gap of H_{pq} and $a = a_i$, $b = b_i$ for some $i \in \{1, \ldots, d\}$. For a suitable choice of $\lambda \in K$ the differential $\omega - \lambda\omega_i$ has greater order than ω. Recursively we obtain that Ω_∞ is generated by the ω_i. It is clear that they are linearly independent. □

Assume that C has the nodes $(x_1, y_1), \ldots, (x_l, y_l)$ and no other singularities, and let H be its Weierstraß semigroup. Consider for $i \in \{1, \ldots, d\}$ the polynomials

$$\Phi_i := X^{a_i}Y^{b_i} + u_{i+1}X^{a_{i+1}}Y^{b_{i+1}} + \cdots + u_d X^{a_d}Y^{b_d}$$

with indeterminates $u_{i+1}, \ldots, u_d$.

Proposition 4.2. *$\gamma_i = (p-1)(q-1)-1-(a_ip+b_iq)$ is a gap of H if and only if the system of linear equations*

$$(G_i) \qquad \Phi_i(x_j, y_j) = 0 \ (j = 1, \ldots, l)$$

has a solution.

Proof. If (G_i) has a solution $(u_{i+1}, \ldots, u_d) \in K^{d-i}$, then Φ_i defines an adjoint of C, and since $a_ip + b_iq$ is the pole order of $\Phi_i(x,y)$ the number γ_i is a gap of H.

Conversely, if γ_i is a gap, there must be a Φ_i of the above shape which defines an adjoint of C, hence (G_i) must have a solution. □

Corollary 4.3. *Let C and C' be nodal curves of type p, q with the Weierstraß semigroups H resp. H' where H has genus g and H' has genus g'. Assume that the nodes of C' form a subset of the set of nodes of C. Then $H' \subset H$. If C has l nodes and C' has l' nodes, then $g' = g + (l - l')$. In particular nodal curves of type p, q with the same nodes have the same Weierstraß semigroup.*

Remark 4.4. If the nodes $(x_1, y_1), \ldots, (x_l, y_l)$ of a nodal curve of type p, q are explicitly known, then Proposition 4.2 can be used to compute the gaps of its Weierstraß semigroup.

As an application of Proposition 4.2 we show

Proposition 4.5. *Let C be a nodal curve of type p, q with $l = l_1 l_2$ nodes $P_{ij} := (x_i, y_j)$ $(i = 0, \ldots, l_1 - 1, j = 0, \ldots, l_2 - 1)$ where $x_0, \ldots, x_{l_1-1}$ resp. $y_0, \ldots, y_{l_2-1}$ are pairwise distinct and*

$$(*) \qquad l_1 \le \frac{q}{2},\ l_2 \le \frac{p}{2}.$$

Then
a) C has the Weierstraß semigroup $H = H_{(l_1-1,l_2-1)}$ corresponding in the sense of Section 1 to the lattice points in the rectangle R_{l_1-1,l_2-1}, i.e.

$$H = \langle p, q, pq - (l_1 p + l_2 q)\rangle \,.$$

b) If (*) is hurt, then no C with such nodes exists.

Proof. Condition (*) implies that the lattice points in the rectangle $R := R_{l_1-1,l_2-1}$ determine the numerical semigroup $H_{(l_1-1,l_2-1)}$ (see Example 1.4a)). If (*) is hurt, then R does not define a semigroup, since the closedness condition of Proposition 1.3 is hurt.

Consider the $l \times l$-matrix $({x_i}^m {y_j}^n)$ given by the P_{ij} and the lattice points $(m, n) \in R$ where we do not require at the moment that (*) is fulfilled. We order its rows and columns as in the proof of Proposition 3.6. Then we get the matrix

$$\Delta := \begin{pmatrix} V & y_0 V & {y_0}^2 V & \dots & {y_0}^{l_2-1} V \\ V & y_1 V & {y_1}^2 V & \dots & {y_1}^{l_2-1} V \\ \vdots & \vdots & \vdots & \dots & \vdots \\ V & y_{l_2-1} V & {y_{l_2-1}}^2 V & \dots & {y_{l_2-1}}^{l_2-1} V \end{pmatrix}$$

with the van der Monde matrix

$$V := \begin{pmatrix} 1 & x_0 & {x_0}^2 & \dots & {x_0}^{l_1-1} \\ 1 & x_1 & {x_1}^2 & \dots & {x_1}^{l_1-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_{l_1-1} & {x_{l_1-1}}^2 & \dots & {x_{l_1-1}}^{l_1-1} \end{pmatrix}.$$

Δ is the Kronecker product of the van der Monde matrix with respect to $y_0, \dots, y_{l_1-1}$ by V.

By elementary column transformations it follows that $\det(\Delta) \neq 0$ (or use the well-known formula $\det(\Delta) = \left(\prod_{l_1-1 \geq i > j \geq 0} (x_i - x_j)\right)^{l_2} \cdot \left(\prod_{l_2-1 \geq i > j \geq 0} (y_i - y_j)\right)^{l_1}$ for Kronecker products). To the lattice point $(l_1 - 1, l_2 - 1) \in R$ corresponds the system of linear equations

$$(G) \qquad {x_j}^{l_1-1} {y_k}^{l_2-1} + \sum_{(\alpha,\,\beta)\in T} u_{\alpha,\,\beta} {x_j}^{\alpha} {y_k}^{\beta} = 0 \quad (j = 0, \dots, l_1 - 1, k = 0, \dots, l_2 - 1)$$

where T is the set of lattice points (α, β) with $(p - 1)(q - 1) - 1 - (\alpha p + \beta q) > \gamma := (p-1)(q-1)-1-((l_1-1)p+(l_2-1)q)$. In particular $R \cap \mathbb{N}^2 \setminus \{(l_1 - 1, l_2 - 1)\} \subset T$.

The coefficient matrix of (G) contains all columns of the Kronecker matrix Δ except for the last one, and it contains for the $(\alpha, \beta) \in T \setminus R$ the columns $s_{\alpha\beta} := ({x_j}^{\alpha} {y_k}^{\beta})$ with $j = 0, \dots, l_1 - 1, k = 0, \dots, l_2 - 1$ where $\alpha < l_1 - 1$ or $\beta < l_2 - 1$. Let $\Delta_{\alpha\beta}$ be the matrix which is obtained from Δ by replacing its last column by $s_{\alpha\beta}$. We want to show that $s_{\alpha\beta}$ is linearly dependent on the other columns of $\Delta_{\alpha\beta}$. By symmetry we may assume that $\alpha < l_1 - 1$. Using $\alpha < l_1 - 1$ we can transform $\Delta_{\alpha\beta}$ by elementary column transformations into a matrix of the form

$$\begin{pmatrix} V & 0 & 0 & \dots & 0 & 0 \\ * & V & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \dots & V & 0 \\ * & * & * & \dots & * & V_\alpha \end{pmatrix}$$

where V_α is obtained from V by replacing its last column by the column $({x_j}^\alpha)_{j=0,\dots,l_1-1}$. Since $\alpha < l_1 - 1$ we have $\det(V_\alpha) = 0$ and hence $\det(\Delta_{\alpha\beta}) = 0$.

It follows that the coefficient matrix of (G) has rank $l-1$ while Δ has rank l. Therefore (G) has no solution. By Proposition 4.2 γ is an element of the Weierstraß semigroup H of C, i.e. $H_{(l_1-1,l_2-1)} \subset H$. As C has exactly l nodes the number of gaps of H is $1/2(p-1)(q-1)-l$, and $H_{(l_1-1,l_2-1)} = H$ follows.

If (*) is hurt we arrive at a contradiction since R_{l_1-1,l_2-1} does not define a numerical semigroup, see Example 1.4a). Thus 4.5b) follows. If (*) is fulfilled H is the semigroup generated by p, q and $\gamma = (p-1)(q-1)-1-((l_1-1)p+(l_2-1)q) = pq-(l_1p+l_2q)$. $\square$

Curves C as in Proposition 4.5 are constructed in Proposition 3.5. Observe that their Weierstraß semigroups depend only on l_1, l_2, and not on the special choice of the x_i ($i = 0,\dots,l_1-1$) and y_j ($j = 0,\dots,l_2-1$). We can choose in particular the set of the l_1l_2 lattice points in the rectangle R_{l_1-1,l_2-1} with the corners $(0,0), (l_1-1,0), (l_1-1,l_2-1), (0,l_2-1)$ as set of nodes. In this case we denote a curve having these nodes by C_{l_1-1,l_2-1}. The Weierstraß semigroup of such a curve is then the numerical semigroup $H_{(l_1-1,l_2-1)}$ determined in the sense of Section 1 by the same rectangle.

It is shown in [W] that all numerical semigroups which are special almost complete intersections are Weierstraß semigroups. In particular all numerical semigroups with 3 generators are so, since by [He] they are complete or special almost complete intersections.

5 Construction of Weierstraß semigroups by elimination of nodes

Let $a := \left\lfloor \frac{q}{2} \right\rfloor, b := \left\lfloor \frac{p}{2} \right\rfloor$. By Example 1.4a) the subsemigroups of $H_{(a-1,b-1)}$ are in one-to-one correspondence with the lattice paths inside $R_{a-1,b-1}$. If $L = (P_0,\dots,P_m)$ is given by such a path, then

$$\alpha_0 = 0 < \alpha_1 < \dots < \alpha_{m-1} \le \alpha_m \le \left\lfloor \frac{q}{2} \right\rfloor - 1, \ \left\lfloor \frac{p}{2} \right\rfloor - 1 \ge \beta_0 \ge \beta_1 > \dots > \beta_{m-1} > \beta_m = 0.$$

Let H_L denote the numerical semigroup corresponding to L. Then

$$H_L = \bigcup_{i=0}^{m} H_{(\alpha_i,\,\beta_i)} \tag{1}$$

(see the second figure in Section 1).

Suppose there exists a nodal curve C_L of type p, q whose nodes are the points of L. Examples for this are the curves C_{l_1-1,l_2-1} with $l_1 \le a, l_2 \le b$ and curves found by elimination of nodes of the C_{l_1-1,l_2-1} (see Proposition 3.6).

Theorem 5.1. *C_L has the Weierstraß semigroup H_L.*

Proof. The statement is true for the rectangles $R_{\alpha_i,\,\beta_i}$ ($i = 0,\dots,m$) (Proposition 4.5). Let H denote the Weierstraß semigroup of C_L. The nodes of $C_{\alpha_i,\,\beta_i}$ form a subset of the set of nodes of C_L, therefore $H_{(\alpha_i,\,\beta_i)} \subset H$ by 4.3 and hence $H_L \subset H$ by (1).

By 2.3 the number of gaps of H is $1/2(p-1)(q-1)-l$ where l is the number of nodes of C_L. But this is also the number of gaps of H_L, hence $H = H_L$. $\square$

An L as in Theorem 5.1 defines a whole series of Weierstraß semigroups:

Corollary 5.2. *Let $p' < q'$ be relatively prime integers with $p < p', q < q'$. Closing the gaps of $H_{p'q'}$ corresponding to the lattice points in L gives a numerical semigroup H'_L. It is the Weierstraß semigroup of a nodal curve of type p', q' with L as its set of nodes.*

Proof. Let $a' := \left\lfloor \frac{p'}{2} \right\rfloor, b' := \left\lfloor \frac{q'}{2} \right\rfloor$. Then $L \subset R_{a-1,b-1} \subset R_{a'-1,b'-1}$ and L defines indeed a numerical subgroup H'_L. Let H be the Weierstraß polynomial defining C_L and Φ that of the curve C' of type p', q' corresponding to $R_{a'-1,b'-1}$. Let $\mathcal{F}'$ be the degree filtration of $K[X,Y]$ with $\deg_{\mathcal{F}}(X) = p', \deg_{\mathcal{F}'}(Y) = q'$ and $\deg_{\mathcal{F}'}(a) = 0$ for $a \in K$. Since $\deg_{\mathcal{F}'}(H) < p'q'$ and the points in L are the singularities of C_L the polynomial H can be used to eliminate the nodes of C' except those in L. We find a nodal curve $C'_L : \Phi + d \cdot H = 0$ of type p', q' which by Theorem 5.1 has the Weierstraß semigroup H'_L. □

Corollary 5.3. *Let L_1 and L_2 be given by lattice paths inside $R_{a-1,b-1}$. Assume that by eliminating the nodes of $C_{a-1,b-1}$ outside of L_1 resp. L_2 we can construct nodal curves C_{L_1} and C_{L_2}. Then $H_{L_1} \cap H_{L_2}$ is a Weierstraß semigroup.*

Proof. This follows from 5.1 since $H_{L_1} \cap H_{L_2} = H_{L_1 \cap L_2}$ and $L_1 \cap L_2$ is by 3.4 the set of nodes of a nodal curve of type p, q obtained from $C_{a-1,b-1}$ by elimination of nodes. □

Example 5.4. For $l_1, l_2 \in \mathbb{N}$ with $1 \le l_1 \le a, 1 \le l_2 \le b$ let L^- be as in Proposition 3.6, and set $x_i = i$ $(i = 0, \dots, l_1 - 1)$, $y_j = j$ $(j = 0, \dots, l_2 - 1)$. Then there is a nodal curve C_{L^-} of type p, q with the set of nodes L^-. Since L^- is given by a lattice path C_{L^-} has the Weierstraß semigroup H_{L^-}. The subsets $\lambda \subset L^-$ given by lattice paths are in one-to-one correspondence with the subsemigroups $H \subset H_{L^-}$ with $p, q \in H$. By Proposition 3.6 all these H are Weierstraß semigroups, because there are curves C_λ with λ as their set of nodes. Similarly for $H_{L^=}$, see Remark 3.7.

For example let $l_1 = a, l_2 = b$ and $r < a$. The set λ of the lattice points on or below the line $X + Y = r - 1$ belongs to L^-, hence H_λ is a Weierstraß semigroup.

Under the weaker assumption that $r < p - 1$ it is shown in [Kn], Satz 5b) that a curve of type p, q exists having only a singularity of multiplicity r with r distinct tangents such that H_λ is the Weierstraß semigroup of this curve.

6 Plane models of type p, q of smooth projective curves

Let $\mathcal{R}$ be a smooth projective curve of genus g and $L = K(\mathcal{R})$ its field of rational (meromorphic) functions over K. For a closed point $P \in \mathcal{R}$ let $\widetilde{\mathcal{R}} := \mathcal{R} \setminus P$ and

$$A := \{f \in L \mid f \in \mathcal{O}_{\mathcal{R},Q} \text{ for all closed } Q \in \widetilde{\mathcal{R}}\}.$$

Then $H(P) = \{-\mathrm{ord}_P(f) \mid f \in A\}$.

Lemma 6.1. *a) Let $x \in A$ with $-\mathrm{ord}_P(x) =: p > 0$ be given. Then A is a free module over $K[x]$ of rank p. In particular A is an affine K-algebra and $K[x] \subset A$ a Noether normalization. Moreover $\widetilde{\mathcal{R}} = \mathrm{Spec}(A)$ is a smooth affine curve and $[L : K(x)] = p$.*
b) Let $\{n_1, \dots, n_t\}$ be the minimal system of generators of $H(P)$ and let $x_i \in A$ with $-\mathrm{ord}_P(x_i) = n_i$ $(i = 1 \dots t)$ be given. Then $A = K[x_1, \dots, x_t]$.

Proof. a) Let h_i be the smallest number in $H(P)$ such that $h_i \equiv i \mod p$ $(i = 0, \ldots, p-1)$. Choose $x_i \in A$ with $-\mathrm{ord}_P(x_i) = h_i$ $(i = 1, \ldots, p-1), x_0 := 1$. For $f \in A$ with $-\mathrm{ord}_P(f) =: \nu$ write $\nu = \mu \cdot p + h_r$ with $\mu \in \mathbb{N}$. Then $x^\mu x_r$ has pole order ν at P and there exists $\lambda \in K$ such that $-\mathrm{ord}_P(f - \lambda x^\mu x_r) < \nu$. Inductively we find elements $\lambda_i \in K$ such that $f - \sum \lambda_i x^\mu x_{r_i}$ has order 0 at P, hence is an element of K. Therefore $\{1, x_1, \ldots, x_{p-1}\}$ is a system of generators of A over $K[x]$. As the functions have different pole orders mod p they are linearly independent.
b) For $f \in A$ write $-\mathrm{ord}_P(f) = \sum_{i=1}^t \mu_i n_i$ $(\mu_i \in \mathbb{N})$. Then there exists $\lambda \in K$ such that $f - \lambda {x_1}^{\mu_1} \cdots {x_t}^{\mu_t}$ has lower pole order at P than f. Inductively we obtain that $f \in K[x_1, \ldots, x_t]$. □

Lemma 6.2. *Let $x, y \in A$ be functions such that $p := -\mathrm{ord}_P(x)$ and $q := -\mathrm{ord}_P(y)$ are relatively prime. Then y is a primitive element of L over $K(x)$ whose minimal polynomial has the form*

$$F(x, Y) = Y^p + f_1(x)Y^{p-1} + \cdots + f_p(x)$$

where $f_1, \ldots, f_p \in K[X]$, $\deg(f_j) \leq \lfloor {}^{jq}/_{p} \rfloor$ $(j = 1, \ldots, p-1)$, $\deg(f_p) = q$. Hence $F(X, Y)$ is a Weierstraß polynomial of type p, q.

Proof. As $K[x]$ is integrally closed in $K(x)$ the minimal polynomial

$$F(x, Y) = Y^s + f_1(x)Y^{s-1} + \cdots + f_s(x)$$

of y has coefficients $f_i(x) \in K[x]$. By Lemma 6.1a) we have $s \leq p$. Since $F(x, y) = 0$ at least two terms of $F(x, y)$ have the same order at P. Assume

$$\deg(f_j) \cdot p + q \cdot (s - j) = \deg(f_l) \cdot p + q \cdot (s - l)$$

with $j, l \in \{0, \ldots, s\}$, $l > j$, where we have set $f_0 = 1$. Then $q \cdot (l - j) \equiv 0 \mod p$. As p and q are relatively prime this is only possible when $s = p$, $l = p$ and $j = 0$. From $[L : K(x)] = p$ we conclude that y is a primitive element of L over $K(x)$.

The orders of the terms $f_1(x)y^{p-1}, \ldots, f_1(x)$ are pairwise distinct, and likewise y^p and $f_j(x)y^{p-j}$ have different order for $j = 1, \ldots, p-1$. We obtain that $\mathrm{ord}_P(y^p) = \mathrm{ord}_P(f_p(x))$ and that y^p and $f_p(x)$ are the terms of lowest order. It follows that $\deg(f_p) = q$ and

$$-\mathrm{ord}_P(f_j(x)y^{p-j}) = \deg(f_j) \cdot p + (p - j) \cdot q < p \cdot q$$

hence $\deg(f_j) \leq \lfloor {}^{jq}/_{p} \rfloor$ for $j = 1, \ldots, p-1$. □

Lemma 6.2 shows that $\mathcal{R}$ has a *plane model* C of type p, q, i.e. that $\mathcal{R}$ is the normalization of the projective closure of C, a fact which is covered by the older literature (see e.g. [Ha] or [HL]). But we want even a plane model which is a nodal curve of type p, q.

Proposition 6.3. *Let $A = K[x_1, \ldots, x_t]$ and let $p, q \in H(P)$ be relatively prime numbers such that*

$$-\mathrm{ord}_P(x_i) < p < q \ (i = 1, \ldots, t).$$

Then there exist elements $u_1, u_2, u_3 \in A$ with

$$-\mathrm{ord}_P(u_1) = p, -\mathrm{ord}_P(u_2) = q, -\mathrm{ord}_P(u_3) < p$$

such that $A = K[u_1, u_2, u_3]$. In particular $C = \mathrm{Spec}(A)$ can be embedded into $\mathbb{A}^3(K)$.

Proof. Choose $u_1 \in A$ with $-\mathrm{ord}_P(u_1) = p$. Then by Lemma 6.1a) $K[u_1]$ is a Noether normalization of A. Choose $y \in A$ with $-\mathrm{ord}_P(y) = q$ and set $u_2 := y + \sum_{i=1}^t a_i x_i$ ($a_i \in K$). Then $-\mathrm{ord}_P(u_2) = q$ and $L = K(u_1, u_2)$ by Lemma 6.2.

If $\mathfrak{m} \in \mathrm{Max}(A)$ is unramified over $K[u_1]$, then $K[u_1]$ contains a local parameter for $\mathfrak{m}$. Now let $\mathfrak{m}$ be ramified and $\tau \in A$ a local parameter for $\mathfrak{m}$. Write

$$\begin{aligned} x_i &= \alpha_{i0} + \alpha_{i1}\tau + \ldots \quad (i = 1, \ldots, t) \\ y &= \beta_0 + \beta_1 \tau + \ldots \end{aligned}$$

($\alpha_{i0}, \alpha_{i1}, \beta_0, \beta_1 \in K$), where the dots represent terms of higher order in τ. At least one $\alpha_{i1} \neq 0$, since the x_i generate A as a K-algebra. We see that $u_2 - (\beta_0 + \sum_{i=1}^t \alpha_{i0} a_i)$ is a local parameter of $\mathfrak{M}$ if $(a_1, \ldots, a_t)$ is chosen in the complement of a hyperplane in $\mathbb{A}^t(K)$. As A is ramified over $K[u_1]$ only at finitely many maximal ideals we can arrange by choosing $(a_1, \ldots, a_t)$ in the complement of finitely many hyperplanes that $K[u_1, u_2]$ contains for every $\mathfrak{m} \in \mathrm{Max}(A)$ a local parameter. Set $B := K[u_1, u_2]$.

We are looking now for a $u_3 \in A$ of the form

$$u_3 = \sum_{i=1}^t b_i x_i, \ (b_1, \ldots, b_t) \in K^t \setminus \{0\}$$

such that different maximal ideals of A are lying over different maximal ideals of $K[u_1, u_2, u_3]$.

Consider points $\mathfrak{M}_1 \neq \mathfrak{M}_2$ on $\mathrm{Max}(A)$. Then

$$\mathfrak{M}_j = (x_1 - \lambda_{j1}, \ldots, x_t - \lambda_{jt}) \ (\lambda_{ji} \in K, \ j = 1, 2, \ i = 1, \ldots, t)$$

with $(\delta_1, \ldots, \delta_t) := (\lambda_{11} - \lambda_{21}, \ldots, \lambda_{1t} - \lambda_{2t}) \neq 0$. If already $\mathfrak{M}_1 \cap B \neq \mathfrak{M}_2 \cap B$, then $\mathfrak{M}_1 \cap K[u_1, u_2, u_3] \neq \mathfrak{M}_2 \cap K[u_1, u_2, u_3]$ for arbitrarily chosen $(b_1, \ldots, b_t) \in K^t$. Otherwise $\mathfrak{M}_1 \cap B = \mathfrak{M}_2 \cap B$ is a singular point of $\mathrm{Max}(B)$. For any choice of $(b_1, \ldots, b_t)$ we have $-\mathrm{ord}_P(u_3) < p$ and

$$\mathfrak{M}_j \cap K[u_1, u_2, u_3] = (u_1 - \bar{u}_1, u_2 - \bar{u}_2, u_3 - \sum_{j=1}^t \lambda_{ij} b_i) \ (j = 1, 2)$$

where $\bar{u}_1, \bar{u}_2 \in K$ do not dependent on j. Then

$$\mathfrak{M}_1 \cap K[u_1, u_2, u_3] \neq \mathfrak{M}_2 \cap K[u_1, u_2, u_3] \text{ if and only if } \sum_{i=1}^t \delta_i b_i \neq 0,$$

i.e. if and only if $(b_1, \ldots, b_t)$ is chosen outside of a hyperplane in K^t. As $\mathrm{Max}(B)$ has only finitely many singularities and over each of them there are only finitely many $\mathfrak{M} \in \mathrm{Max}(A)$ we can choose u_3 as desired.

Set $D := K[u_1, u_2, u_3]$. Then for any $\mathfrak{m} \in \mathrm{Max}(D)$ there is exactly one $\mathfrak{M} \in \mathrm{Max}(A)$ lying over $\mathfrak{m}$. Therefore $A_\mathfrak{m} = A_\mathfrak{M}$ and $A_\mathfrak{m}$ is a finite $D_\mathfrak{m}$-module. Moreover any $\mathfrak{M} \in \mathrm{Max}(A)$ contains a local parameter $\tau \in \mathfrak{m}$. Then $A_\mathfrak{m} = D_\mathfrak{m}$ and consequently $A = D$. □

Theorem 6.4. *Let $p < q$ be relatively prime numbers in $H(P)$ which are greater than the elements of the minimal system of generators of $H(P)$. Then $\mathcal{R}$ has a plane model of type p, q with the place P at infinity which has at most nodes as singularities. In particular every Weierstraß semigroup is the Weierstraß semigroup of a nodal curve of type p, q for properly chosen p, q.*

Proof. Let $\{n_1, \ldots, n_t\}$ be the minimal system of generators of $H(P)$ and let $x_i \in A$ with $-\mathrm{ord}_P(x_i) = n_i$. By Lemma 6.1b) we have $A = K[x_1, \ldots, x_t]$ and with u_1, u_2, u_3 as in Proposition 6.3 we obtain $A = K[u_1, u_2, u_3]$.

It is well-known that the generic projection of $\mathrm{Spec}(A)$ from 3-space into the plane is birationally equivalent to $\mathrm{Spec}(A)$ and has at most nodes as singularities. Its coordinate ring has the form $K[u_1 - a_1u_3, u_2 - a_2u_3]$ where (a_1, a_2) can be chosen in the complement of an algebraic curve in $\mathbb{A}^2(K)$ (see [A], section II of its appendix where also the case characteristic $K > 0$ is settled). Set $u := u_1 - a_1u_3, v := u_2 - a_2u_3$. Then $-\mathrm{ord}_P(u) = p$, $-\mathrm{ord}_P(v) = q$. By Lemma 6.2 the minimal polynomial of v over $K[u]$ is a Weierstraß polynomial of type p, q. Hence $\mathrm{Spec}(K[u, v])$ is a plane model of $\mathcal{R}$ of type p, q having at most nodes as singularities. $\square$

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